**2008 BLUE MOP, FUNCTIONAL EQUATIONS-II**
**ALİ GÜREL**

(1) Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(1) = 2$ and

$$f(xy) = f(x)f(y) - f(x+y) + 1.$$

(2) Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ satisfying

$$f(x+1) = f(x) + 1, \quad \text{and} \quad f(x^3) = f(x)^3 \text{ for all } x \in \mathbb{Q}^+.$$

(3) Find all injective functions $f : \mathbb{N} \to \mathbb{N}$ satisfying, $f(1) = 2$, $f(2) = 4$, and

$$f(f(m) + f(n)) = f(f(m)) + f(n).$$

(4) (BMO-03) Find all possible values for $f\left(\frac{2004}{2003}\right)$ if $f : \mathbb{R} \to [0, +\infty)$ is the function satisfying the conditions:

$$\begin{array}{ll} (i) & f(xy) = f(x)f(y),\ \forall x, y \in \mathbb{Q} \\ (ii) & f(x) \le 1 \Rightarrow f(x+1) \le 1,\ \forall x \in \mathbb{Q} \\ (iii) & f\left(\dfrac{2003}{2002}\right) = 2. \end{array}$$

(5) Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xy) = xf(y) + yf(x).$$

(6) The function $f : \mathbb{R} \to \mathbb{R}$ satisfies $x + f(x) = f(f(x))$ for all real $x$. Find all solutions to the equation $f(f(x)) = 0$.

(7) (SL-79) Given a function $f : \mathbb{R} \to \mathbb{R}$, if for every two real $x, y$ the equality

$$f(xy + x + y) = f(xy) + f(x) + f(y)$$

holds, prove that $f(x+y) = f(x) + f(y)$ for every real numbers $x$ and $y$.

(8) (IMO-90) Construct a function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that

$$f(xf(y)) = \frac{f(x)}{y},\ \forall x, y \in \mathbb{Q}^+.$$

**Problem 1, Solution by Gye Hyun Baek:** Plugging in $x = 0$ gives $f(0) = 1$. Plugging in $y = 1$ gives $f(x + 1) - f(x) = 1$. We conclude that $f(n) = n + 1$ for all integers $n$. Now, letting $y = n$ gives $f(nx) = nf(x) - (n - 1)$. Given $x$, choose $n$ which makes $nx$ an integer. Then $f(nx) = nx + 1$ and we are left with $f(x) = x + 1$ for all rational numbers $x$ $\square$

**Problem 2, Solution by Nicholas Triantafillou:** By induction $f(x + n) = f(x) + n$. Given $x \in \mathbb{Q}^+$, let $n \in \mathbb{Z}^+$ such that both $nx^2$ and $n^2x$ are integers. Then,

$$\begin{aligned} f((x+n)^3) &= f(x^3 + 3x^2n + 3xn^2 + n^3) = f(x)^3 + 3x^2n + 3xn^2 + n^3 \\ &= (f(x) + n)^3 = f(x)^3 + 3f(x)^2n + 3f(x)n^2 + n^3 \end{aligned}$$

implies $f(x) = x$ or $f(x) = -x - n$ but $f(x) > 0$ so we conclude that $f(x) = x$ for all $x \in \mathbb{Q}^+$ $\square$

**Problem 3, Solution by Sam Keller:** Letting $m = 1$ gives $f(f(n)+2) = f(n) + 4$. Since $f(2) = 4$, we deduce that $f(2k) = 2k + 2$ for all positive integers $k$. Now, let $f(3) = c$. Because $f$ is injective, $c$ is odd. For all odd $k > c$, we get $f(k) = k + 2$. So $f(5)$ is either an odd number less than $c$ or it is $c + 2$. If $f(5) = l < c$, then $f(m) = m + 2$ for all odd positive integers $m \geq l$. But $c - 2 \geq l$ is an odd integer so $f(c - 2) = c = f(3)$ and by injectivity we get $c = 5$. However then 5 is in the image of $f$ and so $f(5) = 7$ which is not less than $c = 5$, a contradiction. In the other case, $f(5) = c + 2$, but $c$ is in the image of $f$ so $f(c) = c + 2$ so by injectivity we get $c = 5$. It then follows that $f(1) = 2$ and $f(x) = x + 2$ for all other positive integers $x$ $\square$

**Problem 4, Solution by Zhifan Zhang:** Note that $f(1) = 1$ and $f(\frac{1}{x}) = \frac{1}{f(x)}$. Also $f(x) = f(-x)$. For a non-zero rational number $x$, $f(x) \neq 0$ since otherwise the first condition would force all the images of rationals to be 0, contradicting the third condition. Now, given positive integers $x, y$ using the first two conditions it is not difficult to show that $f(x \pm y) \leq max\{f(x), f(y)\}$. We can generalize this to $f(mx+ny) \leq max\{f(x), f(y)\}$ for all integers $m$ and $n$. Hence, $f(gcd(x, y)) \leq max\{f(x), f(y)\}$. For distinct primes $p$ and $q$ this gives $1 \leq max\{f(p), f(q)\}$. However, $f(1) = 1$ so by the second condition, $f(x) \leq 1$ for all positive integers. Thus $max\{f(p), f(q)\} = 1$ for all distinct primes $p$ and $q$. So all but one of the primes are sent to 1. Factoring 2003 and 2002 and using the third condition we see that this prime is one of 2,7,11, or 13. If it is 2, then we get $f\left(\frac{2004}{2003}\right) = \frac{1}{4}$. In all the other cases $f\left(\frac{2004}{2003}\right) = 1$ $\square$

**Problem 5, Solution by Wenyu Cao:** Letting $x = y = 0$, we get $f(0) = 0$. For $x \neq 0$, let $g(x) = e^{\frac{f(x)}{x}}$. Then our equation becomes $g(xy) = g(x)g(y)$. Since $f$ is continuous, so is $g$ and by Cauchy's Equation, $g(x) = x^c$ for some real $c$, which implies that $f(x) = cx \ln |x|$ with the convention that $0 \ln 0 = 0$ $\square$

**Problem 6, Solution by Matthew Superdock:** Suppose $f(f(a)) = 0$ for some real number $a$. Since $x + f(x) = f(f(x))$ for all real $x$, we have $a + f(a) = 0 \Rightarrow f(a) = -a$ and then $f(f(a)) = 0 \Rightarrow f(-a) = 0$. Hence, we also have $-a + f(-a) = f(f(-a)) \Rightarrow f(0) = -a$. But $0 + f(0) = f(f(0)) \Rightarrow -a = f(-a) = 0$. Therefore $a = 0$, and note that $f(f(0)) = f(0)$ and $f$ is injective so $f(0) = 0$ and $f(f(0)) = 0$, indeed $\square$

**Problem 7, Solution by Minseon Shin:** Plugging-in $(x, y) \to (x, -x)$ in $f(xy+x+y) = f(xy)+f(x)+f(y)$ gives $f(-x) = -f(x)$. So, in particular $f(0) = 0$. Plugging-in $(-x, -y)$ gives $f(xy - x - y) = f(xy) - f(x) - f(y)$ and so $f(xy + x + y) + f(xy - x - y) = 2f(xy)$. When $x = \frac{y}{y-1}$, this becomes $f\left(\frac{2y^2}{y-1}\right) = 2f\left(\frac{y^2}{y-1}\right)$. On the other hand, when $x = \frac{-y}{y+1}$ we get $f\left(\frac{2y^2}{y+1}\right) = 2f\left(\frac{y^2}{y+1}\right)$. Since any real number can be written in one of the forms $\frac{y^2}{y\pm1}$, we conclude that for all real $x$, we have $f(2x) = 2f(x)$.
Plugging-in $(x, y) \to (x, 1)$ in the original equation, we get $f(2x + 1) = 2f(x) + f(1)$. Plugging-in $(x, y) \to (xy + x + y, y)$, we get $f(2xy + 2x + 2y + 1) = 2f(xy) + 2f(x) + 2f(y) + f(1)$. We proceed by rearranging the terms and using the functional equation one more time:

$$\begin{aligned}
f(x(2y+1) + x + (2y+1)) &= 2f(xy) + 2f(x) + 2f(y) + f(1) \Rightarrow \\
f(2xy + x) + f(x) + f(2y+1) &= 2f(xy) + 2f(x) + 2f(y) + f(1) \Rightarrow \\
f(2xy + x) + f(x) + 2f(y) + f(1) &= f(2xy) + 2f(x) + 2f(y) + f(1) \Rightarrow \\
f(2xy + x) &= f(2xy) + f(x)
\end{aligned}$$

Finally plugging-in $y \to \frac{y}{2x}$ above, we get the desired result: $f(x + y) = f(x) + f(y)$ $\square$

**Problem 8, Solution by Joshua Pfeffer:** First observe that $f$ is both injective and surjective. Now, $(x, y) = (1, 1)$ gives $f(1) = 1$. Then, $y = 1$ gives $f(f(x)) = \frac{1}{x}$. Plugging-in $f(y)$ for $y$ we get $f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}$. Equivalently, $f(xy) = f(x)f(y)$. Note that these two conditions $f(xy) = f(x)f(y)$ and $f(f(x)) = \frac{1}{x}$ together imply the original functional equation. Hence, it is enough to construct a multiplicative function satisfying $f(f(x)) = \frac{1}{x}$. Let $p_j$ be the $j$-th prime and for any positive integer $k$ let $f(p_{2k-1}) = p_{2k}$ and $f(p_{2k}) = \frac{1}{p_{2k-1}}$. Extend $f$ to a multiplicative function defined on $\mathbb{Q}^+$ using prime factorization and observe that it satisfies both conditions $\square$